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On Some Topological Properties of Plane Curves and a Theorem of Möbius.[*]

By Solomon Lefschetz.

1. In this paper an application is made of the transformation by inversion to the topology of plane curves. Of the results given many are new, and when they are not, the proofs are. A representative sketch is given for a certain type of closed branches, and is then applied to a proof and generalization of the theorem of Möbius[†] concerning the inflexions of odd branches, proof based upon entirely different premises. We refer to his paper and to others of C. A. Scott[‡] for different applications of the transformation by inversion to the study of plane curves.

2. Let C be a plane branch of curve with a certain number of real points at infinity, and C' its inverse with respect to a pole O exterior to C. As to C', which will have in O a point of multiplicity k and all its points at finite distance, we will assume that it is a *continuous*, *closed*, *rectifiable*[§] branch having only a finite number of multiple points.

If, then, we fix a positive direction on the arc at an arbitrary point of C', such a direction is defined at any other point on any element of arc through it. The number of directions of advancement from any point is even, since to a positive direction must correspond a negative one.

Since an arbitrary circle divides the plane into two separate regions, it follows that it meets C' in an even number of points, for starting from any point of C' we can come back to it after having described the whole branch and must therefore have passed from one region to the other an even number of times. *A circle infinitely near O meets C' in the neighborhood of O, in a number of points the parity of which is independent of the circle.* For, if $O\alpha$ and $O\alpha'$ are two opposite directions of advancement on C' at O, and D is a line through O not tangent to them, then either $O\alpha$ and $O\alpha'$ are on the same side of D (cuspidal arc), or they are separated by it. If now M is a point of intersection of C and Γ near O on $O\alpha$, then, in the first case, to M there corresponds a point M' on $O\alpha'$, and if M disappears, which can happen only if it crosses O, M' also will; while in the second case, if M disappears, M' will appear, and this shows that the number of intersections varies only by a multiple of 2. It follows that the intersections of Γ and C', which are not in the neighborhood of O, have a parity independent of Γ. Passing to the limit, it is found that *the number of points other than O, where the circles through this point meet C', differs only by multiples of* 2. These propositions still hold if we replace Γ everywhere by any closed branch without multiple points. If these variable intersections are in odd (even) number, C is said to be an *odd* (*even*) *branch*, and the inversion gives this: *An odd* (*even*) *branch is met by any line of its plane in an odd* (*even*) *number of points.* The maximum of this number will define the *order* of the branch.

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[*] Presented to the American Mathematical Society, September 11, 1911.

[†] "On the Higher Singularities of a Plane Curve," American Journal of Mathematics, Vol. XIV (1892), p. 301.

[‡] "The Nature and Effect of Singularities of a Plane Curve," *Ibid.*, Vol. XV (1893) p. 221.

[§] For a definition of these words see Camille Jordan, *Traité d'Analyse,* Vol. I, second edition, p. 90.

3. In what follows we will further assume that the branches considered are non-cuspidal, and also that the curvature is continuous, and its first derivative everywhere definite. *In this case C is of even class.* For, when a line OD revolves around a point O, its intersections with C will change, and that by two units, only when OD touches C; but to every appearance of intersection points, there must correspond a disappearance, which shows that the number of contacts is even. *If C is closed, it has an even number of inflexions.* For, when a point M describes the whole branch, the curvature at M, with respect to O, changes an even number of times; but such changes occur only at inflexions, or when OM is tangent in M; and since the class is even, the same holds for the number of inflexions. A closed branch met by any straight line in at most two points is called an *oval.* It can evidently have no multiple points or inflexions, and conversely, *a closed branch without multiple points or inflexions is an oval.* For, if it were met by a straight line in four points, considerations of continuity would show that it has tangents which meet in two more points besides their contact, and then by considering these tangents, it would be found that it has at least one bitangent, hence at least two inflexions.*

4. Let AB be an arc of curve on which the curvature is increasing from A to B (Fig. 1), while α, β are the contacts with the evolute, of the normals at A and B. We have: Chord $\alpha\beta <$ arc $\alpha\beta = \alpha A - \beta B$, which shows that the circle of curvature at B is interior to the one at A.

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* H. G. Zeuthen, "Sur les courbes du 4e ordre," *Math. Ann.*, Vol. VII, p. 412.

A point A at which the curvature is maximum (Fig. 2) or minimum (Fig. 3) will be called a *vertex.** To a point P, on one side of A on the arc, corresponds Q on the other side, such that the curvature is the same at both points. If π, α, κ correspond to P, A, Q on the evolute, and if AP is sufficiently small, we have $O = \pi P - \kappa Q < \pi\kappa < \pi P + \kappa Q$, which shows that *circles of curvature at points separated by A in the neighborhood of A have two real intersections.* Since $\alpha P - \alpha Q$ is very small, αPQ differs but little from an isosceles triangle, $\pi\kappa$ is nearly parallel to PQ, and therefore αA, normal at A, differs but little from the perpendicular bisector of $\pi\kappa$, which is the common chord of the osculating circles at P and Q; hence these two circles have a common point in the neighborhood of A, in the direction of the normal, *that is not on the arc PAQ itself.* At a vertex, the osculating circle is *hyperosculating.* This can be seen from the fact that at such a point the circle is on the same side of the arc on both sides of the contact. We will not go further into this question.

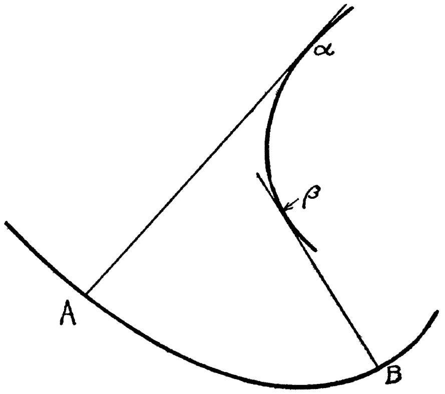

FIG. 1.

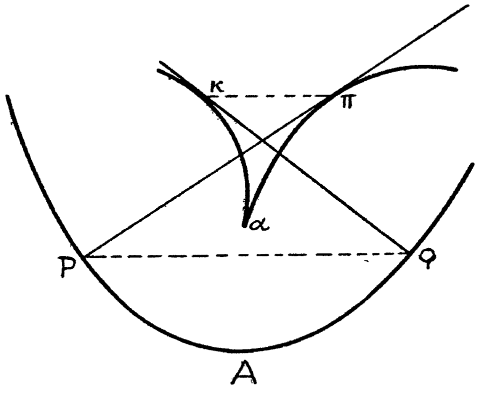

FIG. 2.

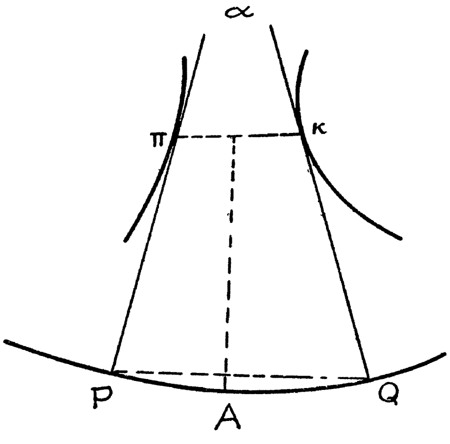

FIG. 3.

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* For an analytic and more rigorous treatment, see: Lilienthal, "Vorlesungen über differentiale Geometrie," Vol. I, p. 56.

5. Since the curvature vanishes at an inflexion, it follows from Rolles' theorem that it has at least one maximum, and the arc at least one vertex, more generally an odd number of vertices between two inflexions. Hence, if the curvature is constantly increasing from A to B (Fig. 1), there is at most one inflexion on AB. If there is none, it follows from § 4 that the osculating circle at A is exterior to the one at B, and that no circle osculating the arc can go through a point interior or exterior to both circles, while one, and one only, passes through any point in the remaining region. If there is one inflexion on the arc, it follows again from § 4 that no circle of curvature passes through a point interior to either circle, and one, and one only, through any point exterior to both. In both cases there are two regions, such that through any point of the one there goes a circle osculating the arc, while none goes through a point of the other.

If the inverse of a closed branch C be taken with respect to a pole M exterior to it, the circles osculating C and going through M become inflexions; hence *their number is even*, and since it changes only when we cross a hyperosculating circle, the proposition is still true if M is on C. Hence *the circles osculating* C *at a point different from a point* M *and going through* M *are in odd number*, and this still holds if M is a point of odd multiplicity. Since C has an even number of inflexions, it follows, from the beginning of this section, that it has also an *even number of vertices.*

6. *A branch* $EA'AE'$ *with two asymptotes* D, D' *which it meets in* A, A', *and with no multiple points, has at least two inflexions* (Fig. 4). For, by reasoning as in § 4, we can find on the arcs $A'E$ and AE' two points F, F' at which the branch has a given curvature arbitrarily small. If we then draw the semi-ellipse Δ hyperosculating the circles of curvatvre Γ, Γ' at F, F' of the given branch, when the curvature at these two points is small enough, we obtain a closed branch, having no multiple points, but met by some straight lines (for example, the base of the semi-ellipse Δ) in four points at least; hence (§ 3) this closed branch is not an oval, and therefore has at least two inflexions, which, since they can not be on the arc $\Gamma\Delta\Gamma'$, are necessarily on the arc $FA'AF'$.

If the branch meets only one of its asymptotes, it has at least one inflexion (Fig. 5). For, by a projection conserving the line at infinity, we may add the condition that the branch is orthogonal to the asymptote which it meets at their intersection A. If we then take the symmetric of the arc AE with respect to D, we can, by a reasoning similar to the preceding, show that there are at least two inflexions on the compound arc, EAE_1, or one at least on the arc AE.

Let us now consider a *loop*, L (Fig. 6), that is an arc of curve having both ends coincident in O. By inverting with respect to O, we obtain a branch such as has just been considered; hence *there are at least one or two circles osculating the loop and going through O, according as one or both circles of curvature at O meet the loop elsewhere.*

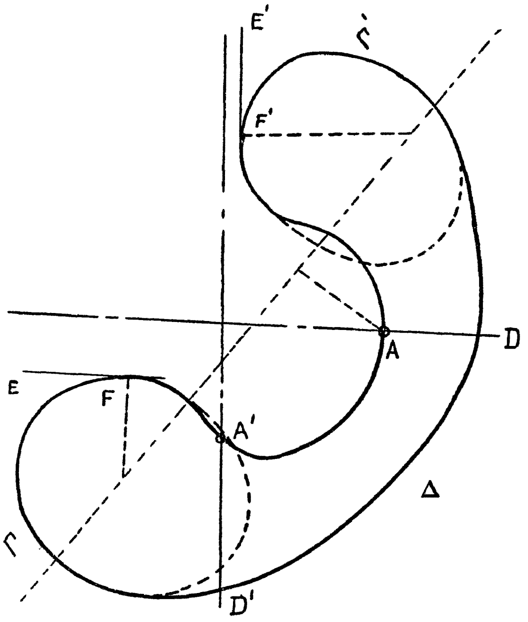


FIG. 4.

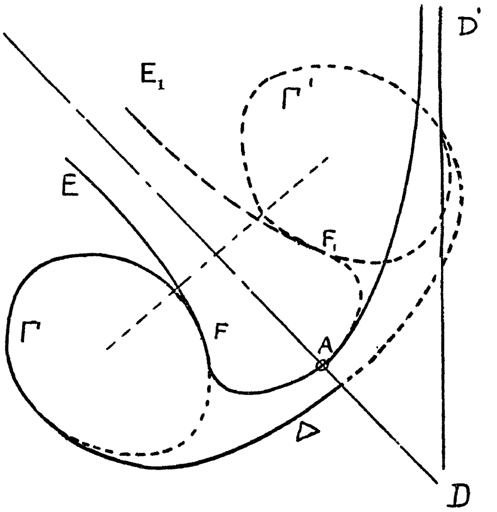


FIG. 5.

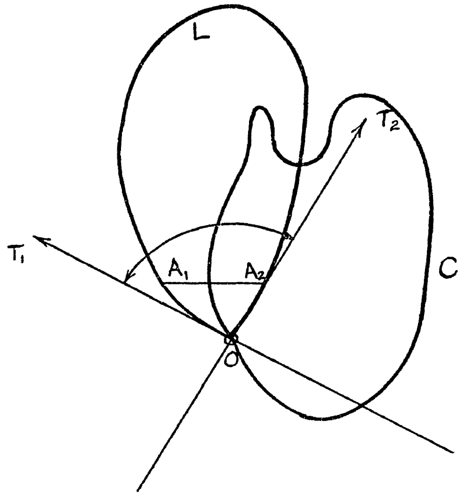


FIG. 6.

Let OT_1 and OT_2 be the semi-tangents to L at O — we will suppose them distinct, although this will not materially affect our conclusions — and let A_1, A_2 be two points in the directions OT_1, OT_2. The arc A_1LA_2 forms, together with the segment A_1A_2, a closed branch without multiple points, and it will be met by any other closed branch C in an even number of points. In particular, *if C goes through O, it will meet L in an odd (even) number of points, if it has in O an odd (even) number of directions of advancement in the angle T_1OT_2.*

7. Given, as in § 2, a branch C, inverse of a closed branch, we can always cut it by a straight line, meeting it in at least two points, passing through none of its multiple points or inflexions, and not touching C. Projecting this line into the line at infinity, a branch C_1 is obtained which will be supposed to have no multiple points, and the inverse C_1' of which will therefore have only one multiple point of order $k \geqslant 2$ at the pole O, while the excess of the order of C on k is even. C_1' is composed of a system of k loops having a common origin O. The loop L_1 will be said to be interior to the loop L_2 (Fig. 7), whenever the angle less than π formed by the directions of advancement on L_1 is interior to the corresponding angle for L_2. O being the only multiple point of C_1', the directions of advancement on one loop can not be separated by those on another; hence two loops are either interior or exterior to each other. A system of loops such as $L_1, L_2, \ldots, L_n$, where L_i is interior to L_{i+1} $(i \leq n-1)$, will be called a *nest* of loops. Counting from L_n each loop will have a certain rank, and the two directions of advancement on L_i will receive the index 1 or 2, when L_i is of odd or even rank.

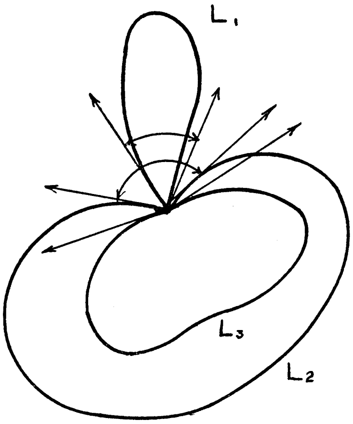


FIG. 7.

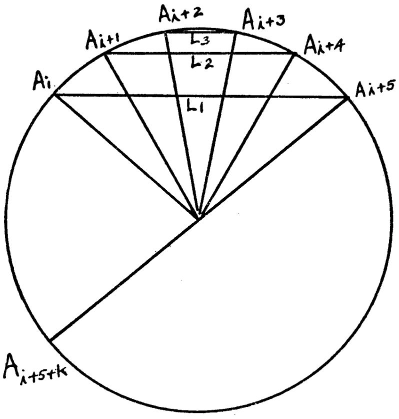
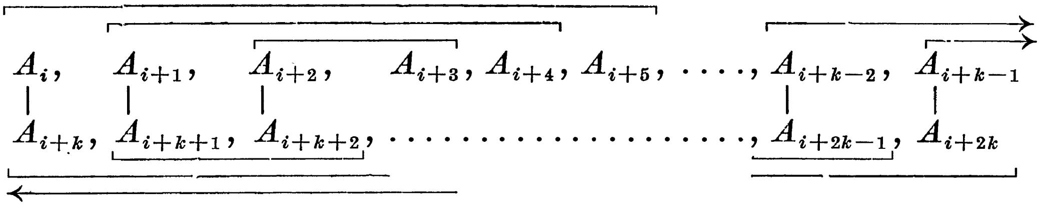


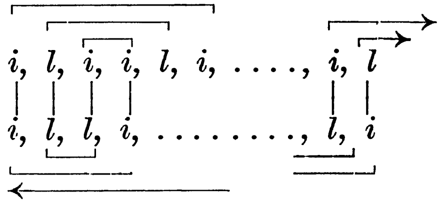
FIG. 8.

8. Let us join the center P of a circle to the vertices $A_1, \ldots, A_{2k}$ of an inscribed regular polygon of $2k$ sides (Fig. 8). The radii will represent, in the order in which they are met, the $2k$ directions of advancement from O. Thus $PA_i, PA_{i+1}, \ldots, PA_{i+5}$ will represent the directions of advancement on

the loops of the nest (L_1, L_2, L_3) in Fig. 7. The loop L_1 will be represented by the triangle OA_iA_{i+5} or by the chord A_iA_{i+5}, etc. We thus obtain a diagram of C_1'. Since the latter is a closed branch, the starred polygon $A_iA_{i+5+k}\ldots$ must be closed. Starting from A_i, we may write on a horizontal line the first k vertices in the order A_i, A_{i+1}, $A_{i+2}, \ldots$, then the vertices A_{i+k+1}, $A_{i+k+2}, \ldots$ on the line below. Indicating then how these vertices are joined, we obtain for C_1' a notation or sketch as follows:


A_i, A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}, A_{i+5}, ..., A_{i+k-2}, A_{i+k-1}
A_{i+k}, A_{i+k+1}, A_{i+k+2}, ..., A_{i+2k-1}, A_{i+2k}

where $A_i = A_j$ if $i \equiv j \pmod{2k}$. More conveniently we may merely write the indices of the corresponding direction of advancement as:



$$\begin{pmatrix} i = 1, 2 \\ l = 1, 2 \\ i \neq l \end{pmatrix}$$

and in both schemes the vertical bars represent the diametral sides of the polygon, while the other sides are represented by the horizontal lines. Since the directions of advancement from O on one loop are never separated by those on another, the chords $A_i\,A_j$ can meet inside of the circle P.

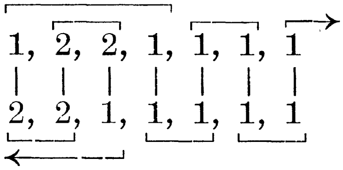
9. Given $A_1, \ldots, A_{2k}$, a starred polygon representing a closed branch C_1' can be obtained from these points in a finite number of ways only. The two properties, given in § 9 will serve to eliminate the polygons corresponding to impossible cases. This will best be shown on a concrete example, and so *let us suppose* $k = 7$. If we represent each combination by its nests and their number of loops, in their order, we may have the following cases:

a) (1, 1, 1, 1, 1, 1, 1), d) (2, 1, 2, 1, 1), g) (3, 2, 1, 1), j) (3, 3, 1),
b) (2, 1, 1, 1, 1, 1), e) (2, 2, 2, 1), h) (3, 1, 2, 1),
c) (2, 2, 1, 1, 1), f) (3, 1, 1, 1, 1), i) (3, 2, 2),

obtained by taking the partitions of 7 in a sum of numbers none greater than 3. The combinations a), d), g), j) alone give a closed polygon, and therefore we will consider only these.

a) *Seven loops* (1, 1, 1, 1, 1, 1, 1). This case is not interesting.

d) *Two nests of two loops and three of one* (2, 1, 2, 1, 1). Fig. 9 gives the corresponding circle diagram, and the sketch is:



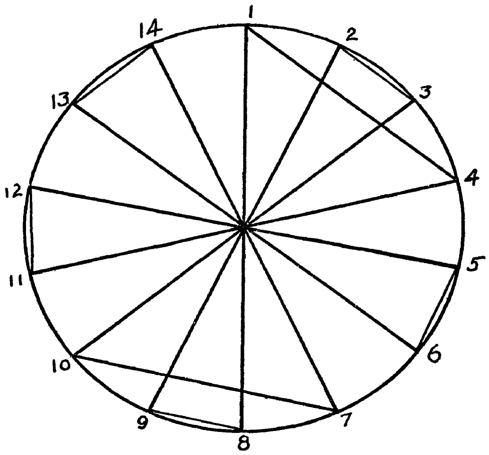
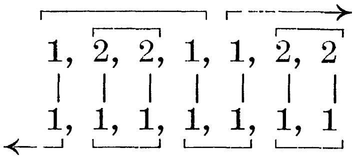


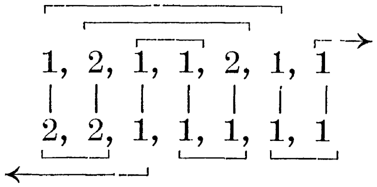
FIG. 9.

It is interesting to notice that the system c) (2, 2, 1, 1, 1) is composite although formed of the same nests as d). In fact its notation is:

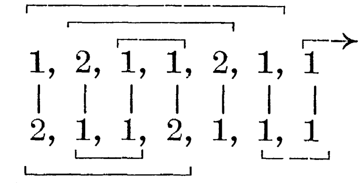


which, as can be seen, does not give a closed polygon.

g) *One nest of three loops, one of two and two of one loop* (3, 2, 1, 1). The notation is:

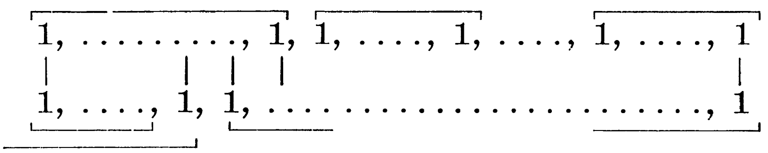


j) *Two nests of three loops and one loop* (3, 3, 1). The notation is:



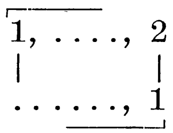
10. We will designate by (i, l) a vertical joining the index i to the index l, in the sketch, independently of the lines on which these indices are found. We may start our sketch from an index corresponding to a loop interior to none. *There are then at least two more couples of consecutive indices* (1) *such as*$\ldots, 1, 1, \ldots$ *than couples of consecutive indices* (2). For all the indices form a certain number of partial systems, each corresponding to a nest of C_1', and both extremities of such a system present a couple $(1, 1)$, while a couple $(2, 2)$ can only be present in the middle of one, and as this can not happen for all middle couples (for then k would be even), our assertion is proved. The dispositions of our sketch can be reduced to two types, according to the relative indices of A_k and A_{k+1}, for both can not have the index 2, as then k would be even. This will be shown when considering case (2).

(1)

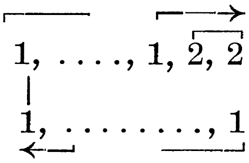


Two verticals $(1, 1)$ are apparent here.

(2)



Each partial system contains an even number of indices, and since k is odd, the same must be true of the number of indices of the partial groups at the end of the first and beginning of the second line; hence we can only have the following disposition for the unwritten indices:



One line $(1, 1)$ is apparent here. We say that there is at least one more. For if not, among the unwritten indices to every couple $(1, 1)$ on one of the lines, there would correspond a couple $(2, 2)$ on the other, so that the couples $(1,1)$ and $(2,2)$ would be in equal number, and we have seen that this can not be if k is odd. Let p, q be the number of bars $(1,1)$ and $(1,2)$ in the sketch. The index (1) will occur $2p + q$ times, and as this number must be even, the same holds for q. Hence the number of bars (i,i) is odd; but it has been found not to be inferior to 2; hence it is at least equal to 3, or: *When k is odd, there are at least three bars (i, i) in the sketch, and their number is always odd.*

11. A circle of curvature $\mathbf{\Gamma}$ of C'_1 at O will be osculating an arc which belongs to two loops A, B. $\mathbf{\Gamma}$ meets C'_1 (Fig. 10) in $k+2$ points coincident with O, hence also in an odd number of points different from O. The k tangents at O being all distinct, it follows from § 6 that the loops interior to A and B are met by $\mathbf{\Gamma}$ in an even number of points different from O, while those exterior to either A or B are met by the circle in an odd number of points. Hence if the ranks of A and B are of the same (opposite) parity, these exterior loops meet $\mathbf{\Gamma}$ in an even (odd) number of points, and the loops A, B meet it in a total odd (even) number of points. Hence if the ranks of A, B are of the same parity, one of the loops A, B, say A, meets $\mathbf{\Gamma}$ in an odd number of points,

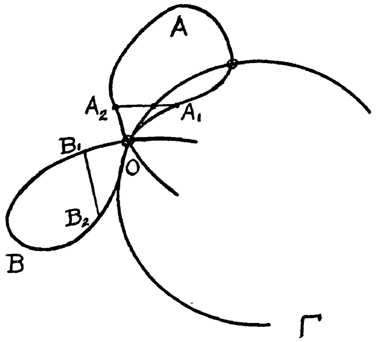


FIG. 10.

and from § 6 we conclude that there is at least one circle osculating A and passing through O. If the circles of curvature for both arcs of A through O meet A, it follows from § 6 that there are at least two circles osculating A and going through O. But if the ranks of A, B are of the same parity, there is a couple (i, i) corresponding to it in the notation for C'_1, and conversely. From § 10 it follows then that *if C'_1 has an odd number of loops there are at least three, and always an odd number, of circles osculating C'_1 at other points than O, and going through O.* By considering C_1 we obtain Möbius' theorem: *An odd branch without multiple points has at least three inflexions.*

12. It will often happen that a minimum greater than 3 can be found for the number of inflexions. Thus, referring to § 9, we can see, in the sketches for C'_1 with seven loops, that in all cases there are at least five lines (i, i), except for the C'_1 represented by $(3, 3, 1)$. Hence there is only one class of odd branches of the seventh order for which no minimum greater than 3 can be found for the inflexions.

The following is a more interesting application: *A spiraloid of order not inferior to* 9 *has at least five inflexions.* For a definition of the spiraloid we refer to Fig. 11. To prove our assertion, it is sufficient to cut the branch

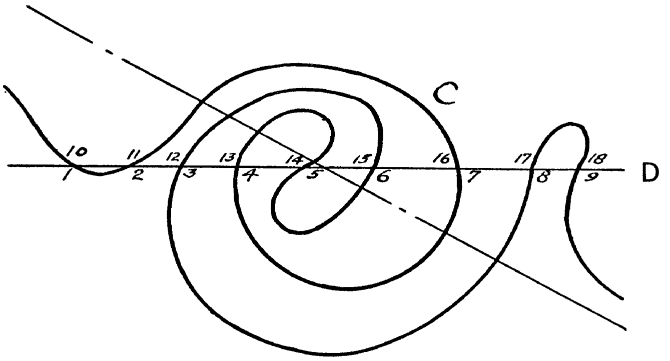
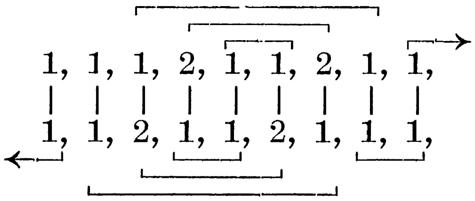


FIG. 11.

by a line D meeting it in at least nine points, to project it into the line at infinity, then to take the inverse C_1' and write its notation, which is found to be the following:



and contains effectively five lines (i, i). This sketch can be easily deduced from C, but we will not enter into any details as to this.

13. From the preceding considerations, minima can in several cases be obtained for the number of vertices of a branch. We have seen in § 5 that no more than one circle of curvature osculating an arc can pass through any point, if there are no vertices on the arc. Hence, from § 5 it follows that *a closed branch with a single* $(2k+1)$*-uple point with* $2k+1$ *distinct tangents has at least* $2k+4$ *vertices*, since there are at least as many osculating circles through O. This supposes $k \geqslant 1$. If $k = 0$ the inverse of the branch with respect to O is an odd branch with one point at infinity, and at least three inflexions, say I, J, K. By reasoning as in § 5, we find that there is at least one vertex in each of the four intervals (∞I), (IJ), (JK), $(K\infty)$, and since inversion in general changes vertices into vertices, the above proposition is also true if $k = 0$. Hence the generalization of a well-known property of ellipses: *An oval has at least four vertices.* In the general case, we obtain by

inversion the following proposition: *An odd branch having* $2k+1$ *distinct points at infinity has at least* $2k+4$ *vertices.* For example, a cubic with only one real asymptote has at least eight vertices: four on the oval and four on the odd branch.

It is well to point out, however, that the oval of a cubic can have six vertices, as, for example:

$$xyz = \lambda\,(x+y+z+\varepsilon x)^3,$$

where λ, ε are sufficiently small and $x+y+z=0$ is the line at infinity which has twelve vertices, six on each branch. A bipartite cubic differing very little from three straight lines has also six vertices on the oval. The cubic $y^2 = 2x(x^2-1)$, with an inflexional tangent at infinity, has six vertices on the oval and three on the odd branch, while one differing very little from it, but with a contact at infinity, may have six vertices on the oval and four on the odd branch. These few examples lay the proper emphasis on the care to be taken in applying the propositions of this section.

Finally, we may remark that we have nowhere supposed our curves to be algebraic, but merely to possess certain definite graphical properties.

WORCESTER, *August* 16, 1911.